Problem: Suppose you had the following data: 8,12,13,15, 15.

Your boss comes to you and asks you to calculate $VaR_{0.95}(X)$ after smoothing your data using a gamma kernel. Further, your gamma kernel has bandwidth parameter $\alpha = 1$.

Answer:
A) 24.5
B) 31.7
C) 38.5
D) 43.4
E) 47.5

Answer:

First let's recall what these items mean:

- $VaR_{0.95}(X) \iff$ Find the x such that $F(x) = 0.95$.
- A kernel acts as a smoothing function. That means that instead of us having just five points, we have five "functions" that we will then weight appropriately.
- Think of it like this: Instead of five points, we have five random variables $X_1, ..., X_5$ distributed via their kernels. Moreover, the mean of each random variable needs to be the same as the observed value. For instance $E(X_1) = \int x \times k_{y_1}(x)dx = y_1$.
- The gamma density with parameters α, θ is as follows:

$$f_{\alpha,\theta}(x) = \frac{(\frac{x}{\theta})^{\alpha} e^{-x/\theta}}{x\Gamma(\alpha)} = \frac{x^{\alpha-1}e^{-x/\theta}}{\theta^{\alpha}\Gamma(\alpha)}$$

Step 0: Let's outline the problem.

- First, we'll calculate the kernel for each of our observations.
- Next, we'll calculate $\hat{f}(x) = \sum p(y_j)k_{y_j}(x)$ where y_j is observation j and k is the kernel.
- We'll integrate out $\hat{f}(x)$ to get $\hat{F}(x)$.
- Finally, we'll solve $\hat{F}(x) = 0.95$.

Step 1: Find $k_y(x)$ for each observation y.

- Recall that the mean of the gamma distribution is simply $\alpha\theta$.
- In our case $\alpha = 1$. Thus the mean is just θ.
- $(E(X) = y) \bigwedge (E(X) = \theta) \rightarrow \theta = y$.
- Plugging these values of α and θ into our density we get:

$$f_{\alpha,\theta}(x) = k_y(x) = \frac{x^{\alpha-1}e^{-x/\theta}}{\theta^\alpha \Gamma(\alpha)} = \frac{e^{-x/y}}{y}$$

Using these facts we can then just plug in each of our observations for y:

$$k_y(x) = \frac{e^{-x/y}}{y} \Rightarrow$$

$$k_8(x) = \frac{e^{-x/8}}{8}$$

$$k_{12}(x) = \frac{e^{-x/12}}{12}$$

$$k_{13}(x) = \frac{e^{-x/13}}{13}$$

$$k_{15}(x) = \frac{e^{-x/15}}{15}$$

Step 2: Now we need to write the density:

$$\begin{aligned}
\hat{f}(x) &= \sum p(y_j)k_{y_j}(x) \\
&= p(8)k_8(x) + p(12)k_{12}(x) + p(13)k_{13}(x) + p(15)k_{15}(x) \\
&= \left(\frac{1}{5}\right)\frac{e^{-x/8}}{8} + \left(\frac{1}{5}\right)\frac{e^{-x/12}}{12} + \left(\frac{1}{5}\right)\frac{e^{-x/13}}{13} + \left(\frac{2}{5}\right)\frac{e^{-x/15}}{15}
\end{aligned}$$

Step 3: We integrate out to get $\hat{F}(x)$:

$$\begin{aligned}
\hat{F}(x) &= \int_0^x \sum p(y_j) k_{y_j}(t) dt \\
&= \int_0^x p(8)k_8(t) + p(12)k_{12}(t) + p(13)k_{13}(t) + p(15)k_{15}(t) dt \\
&= \int_0^x \left(\frac{1}{5}\right) \frac{e^{-t/8}}{8} + \left(\frac{1}{5}\right) \frac{e^{-t/12}}{12} + \left(\frac{1}{5}\right) \frac{e^{-t/13}}{13} + \left(\frac{2}{5}\right) \frac{e^{-t/15}}{15} dt \\
&= \left(\frac{1}{5}\right) \left[(1 - e^{-x/8}) + (1 - e^{-x/12}) + (1 - e^{-x/13}) + 2(1 - e^{-x/15})\right] \\
&= \left(\frac{1}{5}\right) \left[5 - e^{-x/8} - e^{-x/12} - e^{-x/13} - 2e^{-x/15}\right]
\end{aligned}$$

Step 4: Solve $\hat{F}(x) = 0.95$: We want to solve:

$$\left(\frac{1}{5}\right)\left[5 - e^{-x/8} - e^{-x/12} - e^{-x/13} - 2e^{-x/15}\right] = 0.95$$

Clearly, this is hard to do. However, we have 5 possible choices. Simply evaluate every one and you will see that option C, $x = 38.5$ is correct.