Insurance Coverage Modifications

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Table of contents

Review

Review of Y^L and Y^P

A **per-loss random variable** Y^L models the payout for losses X.
A **per-payment random variable** Y^P models the payout for losses X conditional on the payout being positive, i.e. $Y^P = Y^L|(Y^L > 0)$.

Ordinary Deductibles

Definition 5.2

Assume X to be a random variable denoting losses. A policy with an **ordinary deductible** of d pays 0 if $X \leq d$ and pays $X - d$ if $X > d$.

Per Loss

$$Y^L = \begin{cases} 0 & X \leq d \\ X - d & X > d \end{cases} \tag{5.1}$$

Per Payment

$$Y^P = \begin{cases} \text{undefined} & X \leq d \\ X - d & X > d \end{cases} \tag{5.2}$$

Ordinary Deductibles

$$f_{Y^L}(y) = \begin{cases} F_X(d) & y = 0 \\ f_X(y+d) & y > 0 \end{cases} \qquad h_{Y^L}(y) = \begin{cases} \text{undefined} & y = 0 \\ \dfrac{f_X(y+d)}{S_X(y+d)} & y > 0 \end{cases}$$

$$F_{Y^L}(y) = F_X(y+d), \quad y \geq 0 \qquad S_{Y^L}(y) = S_X(y+d), \quad y \geq 0$$

Ordinary Deductibles

$$f_{Y^P}(y) = \begin{cases} 0 & y = 0 \\ \dfrac{f_X(y+d)}{S_X(d)} & y > 0 \end{cases} \qquad h_{Y^P}(y) = \begin{cases} \text{undefined} & y = 0 \\ \dfrac{f_X(y+d)}{S_X(y+d)} & y > 0 \end{cases}$$

$$F_{Y^P}(y) = \frac{F_X(y+d) - F_X(d)}{S_X(d)}, \quad y \geq 0 \qquad S_{Y^P}(y) = \frac{S_X(y+d)}{S_X(d)}, \quad y \geq 0$$

Example

Example 5.3

Let $X \sim \exp(\theta)$. Calculate $f_{Y^P}(y)$ and $f_{Y^L}(y)$.

Example

Answer

$$f_{Y^P}(y) = \frac{f_X(y+d)}{S_X(d)} = \frac{\frac{1}{\theta}e^{-(y+d)/\theta}}{1-F_X(d)} = \frac{\frac{1}{\theta e^{(y+d)/\theta}}}{e^{-d/\theta}} = \frac{1}{\theta e^{y/\theta}} = \frac{1}{\theta}e^{-y/\theta}$$

Observe that $f_{Y^P}(y)$ is exactly the same as $f_X(y)$! This reflects the *memoryless property* of the exponential distribution. Also,

$$f_{Y^L}(y) = f_X(y+d) = \frac{e^{-(y+d)/\theta}}{\theta}$$

Please see Example 5.3 in the textbook for the calculation of the other quantities $S_{Y^P}(y), F_{Y^P}(y), h_{Y^P}(y), S_{Y^L}(y), F_{Y^L}(y)$, and $h_{Y^L}(y)$.

Franchise Deductibles

Definition 5.4

Assume X to be a random variable denoting losses. A policy with a **franchise deductible** of d pays 0 if $X \leq d$ and pays the full loss amount X if $X > d$.

Per Loss

$$Y^L = \begin{cases} 0 & X \leq d \\ X & X > d \end{cases}$$

Per Payment

$$Y^P = \begin{cases} \text{undefined} & X \leq d \\ X & X > d \end{cases}$$

Franchise Deductibles

$$f_{Y^L}(y) = \begin{cases} F_X(d) & y = 0 \\ 0 & 0 < y \leq d \\ f_X(y) & y > d \end{cases} \qquad h_{Y^L}(y) = \begin{cases} 0 & 0 \leq y \leq d \\ h_X(y) & y > d \end{cases}$$

$$F_{Y^L}(y) = \begin{cases} F_X(d) & 0 \leq y \leq d \\ F_X(y) & y > d \end{cases} \qquad S_{Y^L}(y) = \begin{cases} S_X(d) & 0 \leq y \leq d \\ S_X(y) & y > d \end{cases}$$

Franchise Deductibles

$$f_{Y^P}(y) = \begin{cases} 0 & 0 \leq y \leq d \\ \dfrac{f_X(y)}{S_X(d)} & y > d \end{cases}$$

$$h_{Y^P}(y) = \begin{cases} 0 & 0 \leq y \leq d \\ h_X(y) & y > d \end{cases}$$

$$F_{Y^P}(y) = \begin{cases} 0 & 0 \leq y \leq d \\ \dfrac{F_X(y) - F_X(d)}{S_X(d)} & y > d \end{cases}$$

$$S_{Y^P}(y) = \begin{cases} 1 & 0 \leq y \leq d \\ \dfrac{S_X(y)}{S_X(d)} & y > d \end{cases}$$

Expected Values

Theorem 5.5

For an ordinary deductible d,

$$E(Y^L) = E(X) - E(X \wedge d)$$

$$E(Y^P) = \frac{E(X) - E(X \wedge d)}{S_X(d)}$$

For a franchise deductible d,

$$E(Y^L) = E(X) - E(X \wedge d) + d(S_X(d))$$

$$E(Y^P) = \frac{E(X) - E(X \wedge d)}{S_X(d)} + d$$

Example

Example 5.6

Let $X \sim$ exponential($\theta = 1000$) and let $d = 250$. When d is an ordinary deductible, what is the expected cost per loss and expected cost per payment?

Example

Answer

From the formula sheet, we know that $E(X \wedge d) = \theta(1 - e^{-d/\theta})$, so

$$E(Y^L) = E(X) - E(X \wedge d) = \theta - \theta(1 - e^{-\frac{d}{\theta}}) = 1000e^{-\frac{250}{1000}} = 778.801$$

$$E(Y^P) = \frac{E(X) - E(X \wedge d)}{1 - F(d)} = \frac{\theta e^{-\frac{d}{\theta}}}{e^{-\frac{d}{\theta}}} = \theta = 1000$$

Read the full Example 5.6 in the textbook to calculate the same quantities for a franchise deductible.

LER

Definition 5.7

The **loss elimination ratio (LER)** is the percent decrease in the expected payment with a policy modification. For losses X, and per-loss Y^L,

$$LER = \frac{E(X) - E(Y^L)}{E(X)}$$

LER

For an ordinary deductible d, the LER is given by:

$$LER = \frac{E(X) - [E(X) - E(X \wedge d)]}{E(X)} = \frac{E(X \wedge d)}{E(X)}$$

For a franchise deductible d, the LER is given by:

$$\begin{aligned} LER &= \frac{E(X) - [E(X) - E(X \wedge d) + d(1 - F_X(d))]}{E(X)} \\ &= \frac{E(X \wedge d) - d(1 - F_X(d))}{E(X)} \end{aligned}$$

See Example 5.8 in the textbook for an example.

Inflation

Theorem 5.9

Consider X, a random variable denoting losses for the current year. Given an ordinary deductible d after inflation of $1+r$, the expected cost per loss variable for next year is:

$$E(Y^L) = (1+r)\left[E(X) - E\left(X \wedge \frac{d}{1+r}\right)\right] \quad (5.3)$$

If $S_X(\frac{d}{1+r}) > 0$, then, the expected cost per payment for next year is:

$$E(Y^P) = \frac{(1+r)\left[E(X) - E\left(X \wedge \frac{d}{1+r}\right)\right]}{S_X\left(\frac{d}{1+r}\right)} \quad (1)$$

See Example 5.10 in the textbook for an example.

Policy Limit

Definition 5.11

A **policy limit** of u specifies u as the maximum payout, after the application of a deductible.

$$F_Y(y) = \begin{cases} F_X(y), & y < u \\ 1 & y \geq u \end{cases}$$

$$f_Y(y) = \begin{cases} f_X(y), & y < u \\ 1 - F_X(u) & y = u \end{cases}$$

Policy Limit

Theorem 5.12

For a policy limit of u applied after inflation $1+r$, the expected cost per loss is:

$$E(Y^L) = (1+r)E\left(X \wedge \frac{u}{1+r}\right)$$

Example

Example 5.13

Let losses $X \sim$ exponential($\theta = 1000$) and let $r = 10\%$ with $u = 2000$. What is the expected cost per loss? What is the proportional reduction to the case where there is no limit?

Example

Answer

The expected cost per loss is

$$(1+r)E\left(X \wedge \frac{u}{1.1}\right) = 1.1E\left(X \wedge 1818\right) \approx 921.415$$

If there was no limit, then the cost per loss is

$$1.1E(X) = 1.1\theta = 1100$$

Thus, by imposing a policy limit, there is a reduction of

$$\frac{1100 - 921.415}{1100} \approx 16.235\%$$

Coinsurance & Maximum Covered Loss

Definition 5.14

A **coinsurance** of α is the proportion of a loss covered by the insurance company, after applying inflation and the deductible. (The customer pays the rest). Thus, the insurance company is only responsible for αY of each post-deductible/inflation-adjusted loss Y.

Maximum Covered Loss

Definition 5.15

A **maximum covered loss** of u is the largest amount of your losses that will be covered.

The maximum covered loss is not to be confused with the policy limit, also denoted as u. The maximum covered loss is applied *before* the deductible, unlike the policy limit.

In the remainder parts of this section, u will be used to denote maximum covered loss.

All Combined

For losses X, subject to ordinary deductible d, maximum covered loss u, inflation r, and coinsurance α, the per-loss variable is given by

$$Y^L = \begin{cases} 0, & X < \frac{d}{1+r} \\ \alpha[(1+r)X - d] & \frac{d}{1+r} \le X < \frac{u}{1+r} \\ \alpha(u-d) & X \ge \frac{u}{1+r} \end{cases}$$

Theorem 5.16

$$E(Y^L) = \alpha(1+r)\left[E\left(X \wedge \frac{u}{1+r}\right) - E\left(X \wedge \frac{d}{1+r}\right)\right] \quad (5.5)$$

$$E(Y^P) = \frac{E(Y^L)}{S_X(\frac{d}{1+r})} \quad (5.6)$$

Example

Example 5.17

Suppose losses are exponentially distributed with $\theta = 1000$. Let inflation $r = 10\%$, maximum covered loss $u = 2000, d = 250$, and $\alpha = 70\%$. What is the expected cost per loss?

Example

Answer

First, $\frac{u}{1+r} = \frac{2000}{1.1} = 1818.18$ and $\frac{d}{1+r} = \frac{250}{1.1} = 227.27$. Thus,

$$\begin{aligned} E(Y^L) &= \alpha(1+r)(E(X \wedge \frac{u}{1+r}) - E(X \wedge \frac{d}{1+r})) \\ &= (0.7)(1.1)[E(X \wedge 1818.18) - E(X \wedge 227.27)] \\ &= 0.77[837.679 - 203.294] \\ &= 488.48 \end{aligned}$$

Theorem

Theorem 5.18

For losses X, inflation r, maximum covered loss u, coinsurance α, and ordinary deductible d, the second moment of the cost per loss (next year) variable is:

$$\begin{aligned} E[(Y^L)^2] &= \alpha^2(1+r)^2 \Big\{ E\left[(X \wedge \tfrac{u}{1+r})^2\right] \\ &\quad - E\left[(X \wedge \tfrac{d}{1+r})^2\right] - 2(\tfrac{d}{1+r})E(X \wedge \tfrac{u}{1+r}) \qquad (5.7) \\ &\quad + 2(\tfrac{d}{1+r})E(X \wedge \tfrac{d}{1+r}) \Big\} \end{aligned}$$

See Example 5.19 in the textbook.